Efficient Geometric Operations on Polyhedra

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1. Introduction

Convex polyhedra and geometric operations on them play an important role in the fields of hybrid model checking [4], program verification [5], and motion planning. In this extended abstract we consider four geometric operations which are convex hull[1], Minkowski sum, intersection, and linear transformation of convex polyhedra. The Minkowski-Weyl Theorem states that a closed convex polyhedron $\mathbf{P}$—we use the term polyhedron for short—can equivalently be represented as an intersection of closed half-spaces $\mathbf{P}(A, \mathbf{a}) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{a}\}$, the so-called *$\mathcal{H}$-representation*, or as a *$\mathcal{V}$-representation*, that is, the Minkowski sum $\mathbf{P} = \mathrm{cone}(\mathbf{U}) + \mathrm{conv}(\mathbf{V})$ of the finitely generated cone of rays $\mathbf{u} \in \mathbf{U}$ and the finitely generated convex hull of vertices $\mathbf{v} \in \mathbf{V}$. Both representations differ algorithmically [8]: While the convex hull and the Minkowski sum can easily be computed for $\mathcal{V}$-representations, the enumeration of the facets of the convex hull and the Minkowski sum is NP-hard if the polyhedra are given in $\mathcal{H}$-representation, and the contrary holds for the intersection, see Tiwary [7]. The problem of converting between both representations is known as the *vertex enumeration* and *facet enumeration problem*, respectively, and its complexity is still open [1]. Note that, since the output size of these geometric operations and also of the conversion can clearly be exponential in the input size, one typically measures the complexity of *output-sensitive* algorithms in this area.

Verification of hybrid systems by symbolic state-space exploration involves repeated applications of the geometric operations named above. Using either of the $\mathcal{V}$- or $\mathcal{H}$-representation, exact computations of these operations are only possible for lower-dimensional systems. In order to tackle higher-dimensional systems, various techniques have been developed. These techniques are, for instance, (i) replacement of the exact geometric operation by simplified versions, e. g. using the *weak join* or the *inversion join* instead of the convex hull [5]; (ii) usage of other representations for which some of the geometric operations behave nicely, e. g. *zonotopes*, which allow an efficient computation of Minkowski sums [3]; or (iii) the usage of *template polyhedra* which are $\mathcal{H}$-polyhedra $\mathbf{P}(A_{\mathrm{fix}}, \mathbf{a})$ where the representation matrix A_{fix} is fixed a priori [6].

In 2009, Le Guernic and Girard proposed the usage of *support functions* for hybrid model checking [4]. For a—not necessarily convex—set $\mathbf{S} \subseteq \mathbb{R}^d$ and a direction $\mathbf{n} \in \mathbb{R}^d$ the value of the support function is defined as $h_{\mathbf{S}}(\mathbf{n}) = \sup_{\mathbf{x} \in \mathbf{S}} \mathbf{n}^T\mathbf{x}$, which coincides in the case of an $\mathcal{H}$-polyhedron $\mathbf{P}(A, \mathbf{a})$ with the optimal value of the linear program "maximize $\mathbf{n}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{a}$". The support function behaves nicely under most geometric operations; in detail, for any two compact convex sets $\mathbf{P}$ and

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[1] Throughout this paper we use the term convex hull as an abbreviation for closed convex hull.

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$\mathbf{Q}$ in $\mathbb{R}^d$ and any $(d \times d)$-transformation matrix M the following easily computable equations hold:

$$h_{M(\mathbf{P})}(\mathbf{n}) = h_{\mathbf{P}}(M^T\mathbf{n}), \quad h_{\mathbf{P}+\mathbf{Q}}(\mathbf{n}) = h_{\mathbf{P}}(\mathbf{n}) + h_{\mathbf{Q}}(\mathbf{n}),$$
$$h_{\mathrm{conv}(\mathbf{P}\cup\mathbf{Q})} = \max(h_{\mathbf{P}}(\mathbf{n}), h_{\mathbf{Q}}(\mathbf{n})),$$

while the intersection is not easily computable

$$h_{\mathbf{P}\cap\mathbf{Q}}(\mathbf{n}) = \inf_{\mathbf{m}\in\mathbb{R}^d} h_{\mathbf{P}}(\mathbf{n}-\mathbf{m}) + h_{\mathbf{Q}}(\mathbf{m}).$$

Support functions, in combination with template polyhedra to overcome the difficulties with the intersections, have been implemented in SpaceEx, a verification platform for hybrid systems [2].

2. Symbolic Orthogonal Projections

We present a new representation of convex polyhedra which we call symbolic orthogonal projections, or sops, for short. Sops can be realized in any vector space $\mathbb{K}^d$ over an ordered field $\mathbb{K}$. A sop $\mathbf{P} = \mathbf{P}(A, L, \mathbf{a}) \subseteq \mathbb{K}^d$, where A is an $(m \times d)$-matrix, L is an $(m \times k)$-matrix, and $\mathbf{a}$ is a column vector in $\mathbb{K}^m$, is the orthogonal projection of an $\mathcal{H}$-polyhedron $\mathbf{P}\left(\begin{pmatrix}A & L\end{pmatrix}, \mathbf{a}\right) \subseteq \mathbb{K}^{d+k}$ onto $\mathbb{K}^d$, where k is the number of columns in L, i. e.

$$\mathbf{P} = \mathbf{P}(A, L, \mathbf{a}) = \left\{\mathbf{x} \in \mathbb{K}^d \,\middle|\, \exists \mathbf{z} \in \mathbb{K}^k,\ A\mathbf{x} + L\mathbf{z} \le \mathbf{a}\right\}.$$

Obviously, the sop $\mathbf{P}(A, L, \mathbf{a})$ is empty if and only if $\mathbf{P}\left(\begin{pmatrix}A & L\end{pmatrix}, \mathbf{a}\right)$ is empty, and any $\mathcal{H}$-polyhedron $\mathbf{P} = \mathbf{P}(A, \mathbf{a}) \in \mathbb{K}^d$ may be represented by the sop $\mathbf{P}(A, \emptyset, \mathbf{a})$, where $\emptyset$ denotes the empty matrix. Furthermore, for a sop $\mathbf{P}(A, L, \mathbf{a}) \subseteq \mathbb{K}^d$ and any given direction $\mathbf{n} \in \mathbb{K}^d$ the optimal value of the linear program "maximize $\mathbf{n}^T\mathbf{x}$ subject to $A\mathbf{x} + L\mathbf{z} \le \mathbf{a}$" provides the value of the support function $h_{\mathbf{P}}(\mathbf{n})$.

A sop $\mathbf{P}(A, L, \mathbf{a})$ is *complete* if there exists some $\mathbf{u} \ge 0$ with $\mathbf{0} = A^T\mathbf{u}$, $\mathbf{0} = L^T\mathbf{u}$, and $1 = \mathbf{a}^T\mathbf{u}$. Any sop can be completed by adding the redundant row $(\mathbf{0}^T, \mathbf{0}^T, 1)$ to its representation $(A, L, \mathbf{a})$.

Convex Hull, Minkowski Sum, and Intersection. Sops behave nicely under the named geometric operations, as the following proposition shows. In fact, all these operations are realized as block matrices over the original matrices. The zero matrix is denoted by O.

Proposition 1. *Let $\mathbf{P}_1 = \mathbf{P}(A_1, L_1, \mathbf{a}_1)$ and $\mathbf{P}_2 = \mathbf{P}(A_2, L_2, \mathbf{a}_2)$ be two sops in $\mathbb{K}^d$. Then the following equations hold:*

$$\mathrm{conv}(\mathbf{P}_1 \cup \mathbf{P}_2) = \mathbf{P}\left(\begin{pmatrix}A_1\\ \mathrm{O}\end{pmatrix}, \begin{pmatrix}A_1 & L_1 & \mathrm{O} & \mathbf{a}_1\\ -A_2 & \mathrm{O} & L_2 & -\mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1\\ \mathbf{0}\end{pmatrix}\right),\ \mathbf{P}_1, \mathbf{P}_2 \textit{ complete},$$
$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}\left(\begin{pmatrix}A_1\\ \mathrm{O}\end{pmatrix}, \begin{pmatrix}A_1 & L_1 & \mathrm{O}\\ -A_2 & \mathrm{O} & L_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}\right),$$
$$\mathbf{P}_1 \cap \mathbf{P}_2 = \mathbf{P}\left(\begin{pmatrix}A_1\\ A_2\end{pmatrix}, \begin{pmatrix}L_1 & \mathrm{O}\\ \mathrm{O} & L_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}\right).$$

Linear Mappings. Any linear mapping ϕ is uniquely determined by its *transformation matrix* $M \in \mathbb{K}^{n\times m}$, i. e. $\phi(\mathbf{x}) = M\mathbf{x}$. We are interested in the following three types of linear mappings, where the $(n \times n)$-identity matrix is denoted by I_n: (i) *automorphisms*, having invertible transformation matrices; (ii) *orthogonal projections* proj_{d-k}, for $0 \le k \le d$, having matrices of the form $(I_d\ \mathrm{O})$; and (iii) *elementary embeddings* embed_{d+k}, for $k \ge 0$, having matrices of the form $\binom{I_d}{\mathrm{O}}$.

Proposition 2. *Every transformation matrix M can be written as the product $M = S^{-1}EPT^{-1}$, where S and T are invertible, E is the matrix of an elementary embedding, and P is the matrix of an orthogonal projection.*

Proposition 3. *Let $\mathbf{P}_1 = \mathbf{P}(A_1, L_1, \mathbf{a}_1)$ be a sop in $\mathbb{K}^d$, S an invertible $(d\times d)$-transformation matrix of the linear mapping ϕ, proj_{d-k} an orthogonal projection with $0 \le k \le d$, and embed_{d+l} an elementary* *embedding with $l \geq 0$. Then the following equations hold:*

$$\phi(\mathbf{P}_1) = \mathbf{P}(A_1 S^{-1}, L_1, \mathbf{a}_1),$$

$$\mathrm{embed}_{d+l}(\mathbf{P}_1) = \mathbf{P}\left(\begin{pmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & I_k \\ \mathrm{O} & -I_k \end{pmatrix}, \begin{pmatrix} L_1 \\ \mathrm{O} \\ \mathrm{O} \end{pmatrix}, \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}\right),$$

$$\mathrm{proj}_{d-k}(\mathbf{P}_1) = \mathbf{P}(A, L, \mathbf{a}_1),$$

where the matrices A and L are uniquely determined by the stipulation $\begin{pmatrix} A & L \end{pmatrix} = \begin{pmatrix} A_1 & L_1 \end{pmatrix}$ and the demand that A has $d - k$ columns.

3. Conclusion

We have shown that the notion of symbolic orthogonal projections allows to efficiently represent convex hulls, Minkowski sums, intersections, and linear transformations of polyhedra. For all these operations, the complexity is polynomial in the input size; in fact, the first three can be realized in constant time, whereas linear transformations involve a few matrix multiplications and Gaussian elimination. In contrast, for both V- and H-representations some of these operations cannot be performed in polynomial time.

We should address an issue which support functions and sops have in common: Up to now, there is—to the author's best knowledge—no efficient method to decide subset relations or equalities of polyhedra represented as support functions or symbolic orthogonal projections, and it is questionable whether such efficient methods exists.

The following table summarizes the hardness results of performing linear transformations "$M(\cdot)$", Minkowski sum "$\cdot + \cdot$", convex hull "$\mathrm{conv}(\cdot \cup \cdot)$", intersection "$\cdot \cap \cdot$", and deciding subset relations "$\cdot \subseteq \cdot$" on polyhedra in the respective representation, where the plus-sign indicates computability in (weakly) polynomial time and a minus-sign indicates that the enumeration problem is either NP-hard or its complexity is unknown.

Representation	$M(\cdot)$	$\cdot + \cdot$	conv $(\cdot \cup \cdot)$	$\cdot \cap \cdot$	$\cdot \subseteq \cdot$
$\mathcal{V}$-representation	$+$	$+$	$+$	$-$	$+$
$\mathcal{H}$-representation	$+^2$	$-$	$-$	$+$	$+$
support function	$+^3$	$+$	$+$	$-$	$-$
sop	$+$	$+$	$+$	$+$	$-$

[2]for automorphism, [3]for endomorphism

We should note that evaluation of the resulting sops, like emptiness checks or computation of their support functions, involves linear programming. The combination of linear programming and sops is compatible with the usage of support functions in the area of hybrid verification, and, beyond that, it could actually be used as a proper replacement of the support functions. The bearable drawback is that we lose the possibility to describe non-polyhedral, convex sets. On the other hand, we benefit from the underlying $\mathcal{H}$-representation, e. g. we easily find relative interior points or separating hyper-planes of sops.

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